MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

69th Annual

AMC 12A

American Mathematics Competition 12A

Wednesday, February 7, 2018

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (D):** There are currently 36 red balls in the urn. In order for the 36 red balls to represent 72% of the balls in the urn after some blue balls are removed, there must be $36 \div 0.72 = 50$ balls left in the urn. This requires that $100 - 50 = 50$ blue balls be removed.

2. **Answer (C):** The 5-pound rocks have a value of $\$14 \div 5 = \$2.80$ per pound; the 4-pound rocks have a value of $\$11 \div 4 = \$2.75$ per pound; the 1-pound rocks have a value of \$2 per pound. It is not to Carl's advantage to take 1-pound rocks when he can take the larger rocks. Therefore the only issue is how many of the more valuable 5-pound rocks to take, including as many 4-pound rocks as possible in each case. The viable choices are displayed in the following table.

5-pound rocks (\$14 each)	4-pound rocks (\$11 each)	1-pound rocks (\$2 each)	value
3	0	3	\$48
2	2	0	\$50
1	3	1	\$49
0	4	2	\$48

The maximum possible value is \$50.

Note: Although the 5-pound rocks are the most valuable per pound, it was not to Carl's advantage to take as many of them as possible. This situation is an example of the classic knapsack problem for which the so-called "greedy algorithm" is not optimal.

3. **Answer (E):** There are 4 choices for the periods in which the mathematics courses can be taken: periods 1, 3, 5; periods 1, 3, 6; periods 1, 4, 6; and periods 2, 4, 6. Each choice of periods allows $3! = 6$ ways to order the 3 mathematics courses. Therefore there are $4 \cdot 6 = 24$ ways of arranging a schedule.

4. **Answer (D):** Because the statements of Alice, Bob, and Charlie are all incorrect, the actual distance d satisfies $d < 6$, $d > 5$, and $d > 4$. Hence the actual distance lies in the interval $(5, 6)$.

5. **Answer (E):** Factoring $x^2 - 3x + 2$ as $(x-1)(x-2)$ shows that its roots are 1 and 2. If 1 is a root of $x^2 - 5x + k$, then $1^2 - 5 \cdot 1 + k = 0$ and $k = 4$. If 2 is a root of $x^2 - 5x + k$, then $2^2 - 5 \cdot 2 + k = 0$ and $k = 6$. The sum of all possible values of k is $4 + 6 = 10$.

6. **Answer (B):** Note that the given conditions imply that the 6 values are listed in increasing order. Because the median of the these 6 values is n, the mean of the middle two values must be n, so

$$\frac{(m+10)+(n+1)}{2}=n,$$

which implies $m=n-11$. Because the mean of the set is also n,

$$\frac{(n-11)+(n-7)+(n-1)+(n+1)+(n+2)+2n}{6}=n,$$

so $7n-16=6n$ and $n=16$. Then $m=16-11=5$, and the requested sum is $5+16=21$.

7. **Answer (E):** Because $4000=2^5\cdot 5^3$,

$$4000\cdot\left(\frac{2}{5}\right)^n=2^{5+n}\cdot 5^{3-n}.$$

This product will be an integer if and only if both of the factors 2^{5+n} and 5^{3-n} are integers, which happens if and only if both exponents are nonnegative. Therefore the given expression is an integer if and only if $5+n\ge 0$ and $3-n\ge 0$. The solutions are exactly the integers satisfying $-5\le n\le 3$. There are $3-(-5)+1=9$ such values.

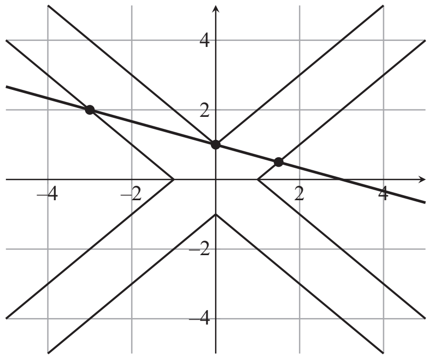
8. **Answer (E):** The length of the base $\overline{DE}$ of $\triangle ADE$ is 4 times the length of the base of a small triangle, so the area of $\triangle ADE$ is $4^2\cdot 1=16$. Therefore the area of $DBCE$ is the area of $\triangle ABC$ minus the area of $\triangle ADE$, which is $40-16=24$.

9. **Answer (E):** If $0\le x\le\pi$ and $0\le y\le\pi$, then $\sin(x)\ge 0$, $\sin(y)\ge 0$, $\cos(x)\le 1$, and $\cos(y)\le 1$. Therefore

$$\sin(x+y)=\sin(x)\cdot\cos(y)+\cos(x)\cdot\sin(y)\le\sin(x)+\sin(y).$$

The given inequality holds for all y such that $0\le y\le\pi$.

10. **Answer (C):** The graph of the system is shown below.



The graph of the first equation is a line with x-intercept $(3, 0)$ and y-intercept $(0, 1)$. To draw the graph of the second equation, consider the equation quadrant by quadrant. In the first quadrant $x > 0$ and $y > 0$, and thus the second equation is equivalent to $|x - y| = 1$, which in turn is equivalent to $y = x \pm 1$. Its graph consists of the rays with endpoints $(0, 1)$ and $(1, 0)$, as shown. In the second quadrant $x < 0$ and $y > 0$. The corresponding graph is the reflection of the first quadrant graph across the y-axis. The rest of the graph can be sketched by further reflections of the first-quadrant graph across the coordinate axes, resulting in the figure shown. There are 3 intersection points: $(-3, 2)$, $(0, 1)$, and $\left(\frac{3}{2}, \frac{1}{2}\right)$, as shown.

OR

The second equation implies that $x = y \pm 1$ or $x = -y \pm 1$. There are four cases:

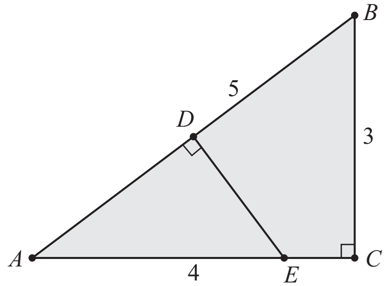
- If $x = y + 1$, then $(y + 1) + 3y = 3$, so $(x, y) = \left(\frac{3}{2}, \frac{1}{2}\right)$.
- If $x = y - 1$, then $(y - 1) + 3y = 3$, so $(x, y) = (0, 1)$.
- If $x = -y + 1$, then $(-y + 1) + 3y = 3$, so again $(x, y) = (0, 1)$.
- If $x = -y - 1$, then $(-y - 1) + 3y = 3$, so $(x, y) = (-3, 2)$.

It may be checked that each of these ordered pairs actually satisfies the given equations, so the total number of solutions is 3.

11. **Answer (D):** The paper's long edge $\overline{AB}$ is the hypotenuse of right triangle ACB, and the crease lies along the perpendicular bisector of $\overline{AB}$. Because $AC > BC$, the crease hits $\overline{AC}$ rather than $\overline{BC}$. Let D be the midpoint of $\overline{AB}$, and let E be the intersection of $\overline{AC}$ and the line through D perpendicular to $\overline{AB}$. Then the crease in the paper

is $\overline{DE}$. Because $\triangle ADE \sim \triangle ACB$, it follows that $\frac{DE}{AD} = \frac{CB}{AC} = \frac{3}{4}$. Thus

$$DE = AD \cdot \frac{CB}{AC} = \frac{5}{2} \cdot \frac{3}{4} = \frac{15}{8}.$$



12. **Answer (C):** If $1 \in S$, then S can have only 1 element, not 6 elements. If 2 is the least element of S, then 2, 3, 5, 7, 9, and 11 are available to be in S, but 3 and 9 cannot both be in S, so the largest possible size of S is 5. If 3 is the least element, then 3, 4, 5, 7, 8, 10, and 11 are available, but at most one of 4 and 8 can be in S and at most one of 5 and 10 can be in S, so again S has size at most 5. The set $S = \{4, 6, 7, 9, 10, 11\}$ has the required property, so 4 is the least possible element of S.

OR

At most one integer can be selected for S from each of the following 6 groups: $\{1, 2, 4, 8\}$, $\{3, 6, 12\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$, and $\{11\}$. Because S consists of 6 distinct integers, exactly one integer must be selected from each of these 6 groups. Therefore 7, 9, and 11 must be in S. Because 9 is in S, 3 must not be in S. This implies that either 6 or 12 must be selected from the second group, so neither 1 nor 2 can be selected from the first group. If 4 is selected from the first group, the collection of integers $\{4, 5, 6, 7, 9, 11\}$ is one possibility for the set S. Therefore 4 is the least possible element of S.

Note: The two collections given in the solutions are the only ones with least element 4 that have the given property. This problem is a special case of the following result of Paul Erdős from the 1930s: Given n integers $a_1, a_2, \ldots, a_n$, no one of them dividing any other, with $a_1 < a_2 < \cdots < a_n \le 2n$, let the integer k be determined by the inequalities $3^k < 2n < 3^{k+1}$. Then $a_1 \ge 2^k$, and this bound is sharp.

13. **Answer (D):** Let S be the set of integers, both negative and nonnegative, having the given form. Increasing the value of a_i by 1 for $0 \le i \le 7$ creates a one-to-one correspondence between S and the ternary (base 3) representation of the integers from 0 through $3^8 - 1$, so S contains $3^8 = 6561$ elements. One of those is 0, and by symmetry, half of the others are positive, so S contains $1 + \frac{1}{2} \cdot (6561 - 1) = 3281$ elements.

OR

First note that if an integer N can be written in this form, then $N - 1$ can also be written in this form as long as not all the a_i in the representation of N are equal to -1. A procedure to alter the representation of N so that it will represent $N - 1$ instead is to find the least value of i such that $a_i \ne -1$, reduce the value of that a_i by 1, and set $a_i = 1$ for all lower values of i. By the formula for the sum of a finite geometric series, the greatest integer that can be written in the given form is

$$\frac{3^8 - 1}{3 - 1} = 3280.$$

Therefore, 3281 nonnegative integers can be written in this form, namely all the integers from 0 through 3280, inclusive. (The negative integers from -3280 through -1 can also be written in this way.)

OR

Think of the indicated sum as an expansion in base 3 using "digits" -1, 0, and 1. Note that the leftmost digit a_k of any positive integer that can be written in this form cannot be negative and therefore must be 1. Then there are 3 choices for each of the remaining k digits to the right of a_k, resulting in 3^k positive integers that can be written in the indicated form. Thus there are

$$\sum_{k=0}^{7} 3^k = \frac{3^8 - 1}{3 - 1} = 3280$$

positive numbers of the indicated form. Because 0 can also be written in this form, the number of nonnegative integers that can be written in the indicated form is 3281.

14. **Answer (D):** By the change-of-base formula, the given equation is equivalent to

$$\begin{aligned}
\frac{\log 4}{\log 3x} &= \frac{\log 8}{\log 2x} \\
\frac{2\log 2}{\log 3 + \log x} &= \frac{3\log 2}{\log 2 + \log x} \\
2\log 2 + 2\log x &= 3\log 3 + 3\log x \\
\log x &= 2\log 2 - 3\log 3 \\
\log x &= \log \frac{4}{27}.
\end{aligned}$$

Therefore $x = \frac{4}{27}$, and the requested sum is $4 + 27 = 31$.

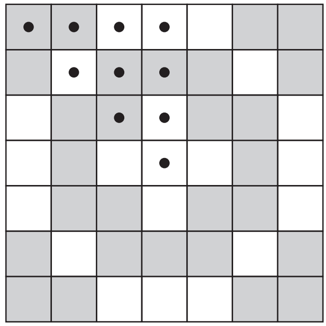
OR

Changing to base-2 logarithms transforms the given equation into

$$\begin{aligned}
\frac{2}{\log_2 3x} &= \frac{3}{\log_2 2x} \\
2\log_2 2x &= 3\log_2 3x \\
\log_2 (2x)^2 &= \log_2 (3x)^3 \\
(2x)^2 &= (3x)^3,
\end{aligned}$$

so $x = \frac{4}{27}$, and the requested sum is $4 + 27 = 31$.

15. **Answer (B):** None of the squares that are marked with dots in the sample scanning code shown below can be mapped to any other marked square by reflections or non-identity rotations. Therefore these 10 squares can be arbitrarily colored black or white in a symmetric scanning code, with the exception of "all black" and "all white". On the other hand, reflections or rotations will map these squares to all the other squares in the scanning code, so once these 10 colors are specified, the symmetric scanning code is completely determined. Thus there are $2^{10} - 2 = 1022$ symmetric scanning codes.



OR

The diagram below shows the orbits of each square under rotations and reflections. Because the scanning code must look the same under these transformations, all squares in the same orbit must get the same color, but one is free to choose the color for each orbit, except for the choice of "all black" and "all white". Because there are 10 orbits, there are $2^{10} - 2 = 1022$ symmetric scanning codes.

A	B	C	D	C	B	A
B	E	F	G	F	E	B
C	F	H	I	H	F	C
D	G	I	J	I	G	D
C	F	H	I	H	F	C
B	E	F	G	F	E	B
A	B	C	D	C	B	A

16. **Answer (E):** Solving the second equation for x^2 gives $x^2 = y + a$, and substituting into the first equation gives $y^2 + y + (a - a^2) = 0$. The polynomial in y can be factored as $(y + (1 - a))(y + a)$, so the solutions are $y = a - 1$ and $y = -a$. (Alternatively, the solutions can be obtained using the quadratic formula.) The corresponding equations for x are $x^2 = 2a - 1$ and $x^2 = 0$. The second equation always has the solution $x = 0$, corresponding to the point of tangency at the vertex of the parabola $y = x^2 - a$. The first equation has 2 solutions if and only if $a > \frac{1}{2}$, corresponding to the 2 symmetric intersection points of the parabola with the circle. Thus the two curves intersect at 3 points if and only if $a > \frac{1}{2}$.

OR

Substituting the value for y from the second equation into the first equation yields

$$x^2 + \left(x^2 - a\right)^2 = a^2,$$

which is equivalent to

$$x^2\left(x^2 - (2a-1)\right) = 0.$$

The first factor gives the solution $x = 0$, and the second factor gives 2 other solutions if $a > \frac{1}{2}$ and no other solutions if $a \le \frac{1}{2}$. Thus there are 3 solutions if and only if $a > \frac{1}{2}$.

17. **Answer (D):** Let the triangle's vertices in the coordinate plane be $(4,0)$, $(0,3)$, and $(0,0)$, with $[0,s]\times[0,s]$ representing the unplanted portion of the field. The equation of the hypotenuse is $3x+4y-12=0$, so the distance from (s,s), the corner of S closest to the hypotenuse, to this line is given by

$$\frac{|3s+4s-12|}{\sqrt{3^2+4^2}}.$$

Setting this equal to 2 and solving for s gives $s = \frac{22}{7}$ and $s = \frac{2}{7}$, and the former is rejected because the square must lie within the triangle. The unplanted area is thus $\left(\frac{2}{7}\right)^2 = \frac{4}{49}$, and the requested fraction is

$$1 - \frac{\frac{4}{49}}{\frac{1}{2}\cdot 4\cdot 3} = \frac{145}{147}.$$

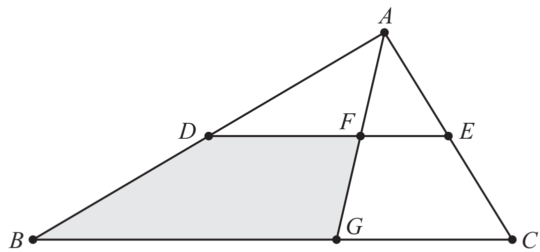
OR

Let the given triangle be described as $\triangle ABC$ with the right angle at B and $AB = 3$. Let D be the vertex of the square that is in the interior of the triangle, and let s be the edge length of the square. Then two sides of the square along with line segments $\overline{AD}$ and $\overline{CD}$ decompose $\triangle ABC$ into four regions. These regions are a triangle with base 5 and height 2, the unplanted square with side s, a right triangle with legs s and $3-s$, and a right triangle with legs s and $4-s$. The sum of the areas of these four regions is

$$\frac{1}{2}\cdot 5\cdot 2 + s^2 + \frac{1}{2}s(3-s) + \frac{1}{2}s(4-s) = 5 + \frac{7}{2}s,$$

and the area of $\triangle ABC$ is 6. Solving $5 + \frac{7}{2}s = 6$ for s gives $s = \frac{2}{7}$, and the solution concludes as above.

18. **Answer (D):** Because AB is $\frac{5}{6}$ of $AB+AC$, it follows from the Angle Bisector Theorem that DF is $\frac{5}{6}$ of DE, and BG is $\frac{5}{6}$ of BC. Because trapezoids $FDBG$ and $EDBC$ have the same height, the area of $FDBG$ is $\frac{5}{6}$ of the area of $EDBC$. Furthermore, the area of $\triangle ADE$ is $\frac{1}{4}$ of the area of $\triangle ABC$, so its area is 30, and the area of trapezoid $EDBC$ is $120-30=90$. Therefore the area of quadrilateral $FDBG$ is $\frac{5}{6}\cdot 90=75$.



Note: The figure (not drawn to scale) shows the situation in which $\angle ACB$ is acute. In this case $BC\approx 59.0$ and $\angle BAC\approx 151^\circ$. It is also possible for $\angle ACB$ to be obtuse, with $BC\approx 41.5$ and $\angle BAC\approx 29^\circ$. These values can be calculated using the Law of Cosines and the sine formula for area.

19. **Answer (C):** Elements of set A are of the form $2^i\cdot 3^j\cdot 5^k$ for nonnegative integers i, j, and k. Note that the product

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right)\left(1+\frac{1}{5}+\frac{1}{5^2}+\cdots\right)$$

will produce the desired sum. By the formula for infinite geometric series, this product evaluates to

$$\frac{1}{1-\frac{1}{2}}\cdot\frac{1}{1-\frac{1}{3}}\cdot\frac{1}{1-\frac{1}{5}}=2\cdot\frac{3}{2}\cdot\frac{5}{4}=\frac{15}{4}.$$

The requested sum is $15+4=19$.

20. **Answer (D):** It follows from the Pythagorean Theorem that $CM=MB=\frac{3}{2}\sqrt{2}$. Because quadrilateral $AIME$ is cyclic, opposite angles are supplementary and thus $\angle IME$ is a right angle. Let $x=CI$ and $y=BE$; then $AI=3-x$ and $AE=3-y$. By the Law of Cosines in $\triangle MCI$,

$$IM^2=x^2+\left(\frac{3}{2}\sqrt{2}\right)^2-2\cdot x\cdot\frac{3}{2}\sqrt{2}\cdot\cos 45^\circ=x^2-3x+\frac{9}{2}.$$

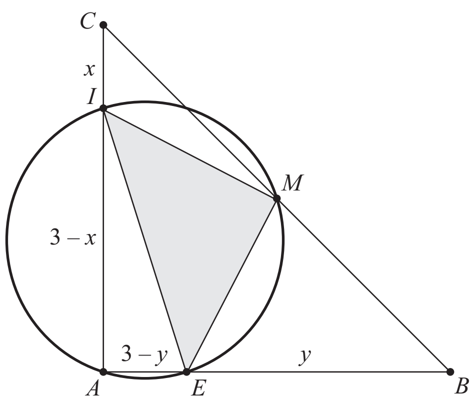
Similarly, $ME^2 = y^2 - 3y + \frac{9}{2}$. By the Pythagorean Theorem in right triangles EMI and IAE,

$$\left(x^2 - 3x + \frac{9}{2}\right) + \left(y^2 - 3y + \frac{9}{2}\right) = (3-x)^2 + (3-y)^2,$$

which simplifies to $x + y = 3$. Because the area of $\triangle EMI$ is 2, it follows that $IM^2 \cdot ME^2 = 16$. Therefore

$$\left(x^2 - 3x + \frac{9}{2}\right)\left((3-x)^2 - 3(3-x) + \frac{9}{2}\right) = 16,$$

which simplifies to $\left(x^2 - 3x + \frac{9}{2}\right)^2 = 16$. Because $y > x$, the only real solution is $x = \frac{3-\sqrt{7}}{2}$. The requested sum is $3 + 7 + 2 = 12$.



OR

Place the figure in the coordinate plane with A at $(0,0)$, B at $(3,0)$, and C at $(0,3)$. Then M is at $\left(\frac{3}{2}, \frac{3}{2}\right)$. Let $s = AE$ and $t = CI$. Then the coordinates of E are $(s, 0)$, and the coordinates of I are $(0, 3-t)$. Because $AIME$ is a cyclic quadrilateral and $\angle EAI$ is a right angle, $\angle IME$ is a right angle. Therefore $\overline{MI}$ and $\overline{ME}$ are perpendicular, so the product of their slopes is

$$\frac{\frac{3}{2}}{\frac{3}{2} - s} \cdot \frac{t - \frac{3}{2}}{\frac{3}{2}} = -1;$$

this equation simplifies to $s = t$. Then, with brackets indicating area,

$$\begin{aligned}[ABC] &= [CIM] + [BME] + [AEI] + [IME] \\ \frac{9}{2} &= \frac{1}{2} \cdot \frac{3}{2} \cdot t + \frac{1}{2} \cdot \frac{3}{2} \cdot (3-t) + \frac{1}{2} \cdot t \cdot (3-t) + 2,\end{aligned}$$

which simplifiles to $2t^2 - 6t + 1 = 0$. Therefore $t = \frac{3 \pm \sqrt{7}}{2}$, and because $AI > AE$, the length of CI is $\frac{3-\sqrt{7}}{2}$ and the requested sum is $3 + 7 + 2 = 12$.

21. **Answer (B):** By Descartes' Rule of Signs, none of these polynomials has a positive root, and each one has exactly one negative root. Because each polynomial is positive at $x = 0$ and negative at $x = -1$, it follows that each has exactly one root between -1 and 0. Note also that each polynomial is increasing throughout the interval $(-1, 0)$. Because $x^{19} > x^{17}$ for all x in the interval $(-1, 0)$, it follows that the polynomial in choice **A** is greater than the polynomial in choice **B** on that interval, which implies that the root of the polynomial in choice **A** is less than the root of the polynomial in choice **B**. Because $x^{13} > x^{11}$ for all x in the interval $(-1, 0)$, it follows that the polynomial in choice **C** is greater than the polynomial in choice **A** on that interval, which implies that the root of the polynomial in choice **C** is less than the root of the polynomial in choice **A** and therefore less than the root of the polynomial in choice **B**. The same reasoning shows that the root of the polynomial in choice **D** is less than the root of the polynomial in choice **B**.

    Furthermore, $2018 > 2018x^6$ on the interval $(-1, 0)$, so $x^6 + 2018 > 2019x^6$, from which it follows that $x^{11}(x^6 + 2018) < 2019x^{17}$. Therefore the polynomial in choice **B** is less than $2019x^{17} + 1$ on the interval $(-1, 0)$. The polynomial in choice **E** has root $-\left(1 - \frac{1}{2019}\right)$. Bernoulli's Inequality shows that $(1 + x)^{17} > 1 + 17x$ for all $x > -1$, which implies that

$$-2019\left(1 - \frac{1}{2019}\right)^{17} + 1 < -2019\left(1 - \frac{17}{2019}\right) + 1 = -2001 < 0,$$

    so the polynomial in choice **B** is negative at the root of the polynomial in choice **E**. This shows that the root of the polynomial in choice **B** is greater than the root in choice **E**.

    Because the unique real root of the polynomial in choice **B** is greater than the unique root of the polynomial in each of the other choices, that polynomial has the greatest real root.

22. **Answer (A):** Let $z = a + bi$ be a solution of the first equation, where a and b are real numbers. Then $(a + bi)^2 = 4 + 4\sqrt{15}i$. Expanding the left-hand side and equating real and imaginary parts yields

$$a^2 - b^2 = 4 \quad \text{and} \quad 2ab = 4\sqrt{15}.$$

From the second equation, $b = \frac{2\sqrt{15}}{a}$, and substituting this into the first equation and simplifying gives $(a^2)^2 - 4a^2 - 60 = 0$, which factors as $(a^2 - 10)(a^2 + 6) = 0$. Because a is real, it follows that $a = \pm\sqrt{10}$, from which it then follows that $b = \pm\sqrt{6}$. Thus two vertices of the parallelogram are $\sqrt{10} + \sqrt{6}i$ and $-\sqrt{10} - \sqrt{6}i$. A similar calculation with the other given equation shows that the other two vertices of the parallelogram are $\sqrt{3} + i$ and $-\sqrt{3} - i$. The area of this parallogram can be computed using the shoelace formula, which gives the area of a polygon in terms of the coordinates of its vertices (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) in clockwise or counter-clockwise order:

$$\frac{1}{2} \cdot \Big|(x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) \\ - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1)\Big|.$$

In this case $x_1 = \sqrt{10}$, $y_1 = \sqrt{6}$, $x_2 = \sqrt{3}$, $y_2 = 1$, $x_3 = -\sqrt{10}$, $y_3 = -\sqrt{6}$, $x_4 = -\sqrt{3}$, and $y_4 = -1$. The area is $6\sqrt{2} - 2\sqrt{10}$, and the requested sum of the four positive integers in this expression is 20.

OR

The solutions of $z^2 = 4 + 4\sqrt{15}i = 16 \operatorname{cis} 2\theta_1$ are $z_1 = 4 \operatorname{cis} \theta_1$ and its opposite, with $0 < \theta_1 < \frac{\pi}{4}$ and $\tan 2\theta_1 = \sqrt{15}$. Then $\cos 2\theta_1 = \frac{1}{4}$, and by the half-angle identities, $\cos\theta_1 = \frac{\sqrt{10}}{4}$ and $\sin\theta_1 = \frac{\sqrt{6}}{4}$. Similarly, the solutions of $z^2 = 2 + 2\sqrt{3}i = 4 \operatorname{cis} \theta_2$ are $z_2 = 2 \operatorname{cis} \theta_2$ and its opposite, with $0 < \theta_2 < \frac{\pi}{4}$ and $\tan 2\theta_2 = \sqrt{3}$. Then $\cos\theta_2 = \frac{\sqrt{3}}{2}$ and $\sin\theta_2 = \frac{1}{2}$.

The area of the parallelogram in the complex plane with vertices z_1, z_2, and their opposites is 4 times the area of the triangle with vertices 0, z_1, and z_2, and because the area of a triangle is one-half the product of the lengths of two of its sides and the sine of their included angle, it follows that the area of the parallelogram is

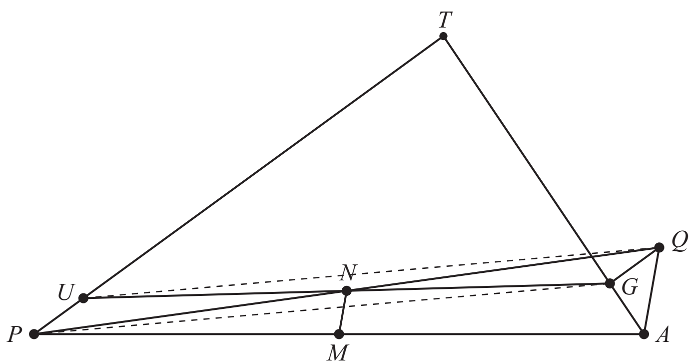
$$\begin{aligned} 4\left(\frac{1}{2} \cdot 4 \cdot 2 \cdot \sin(\theta_1 - \theta_2)\right) &= 16(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) \\ &= 16\left(\frac{\sqrt{6}}{4} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{10}}{4} \cdot \frac{1}{2}\right) \\ &= 6\sqrt{2} - 2\sqrt{10}. \end{aligned}$$

Therefore, $p + q + r + s = 6 + 2 + 2 + 10 = 20$.

23. **Answer (E):** Extend $\overline{PN}$ through N to Q so that $PN = NQ$. Segments $\overline{UG}$ and $\overline{PQ}$ bisect each other, implying that $UPGQ$ is a parallelogram. Therefore $\overline{GQ} \parallel \overline{PT}$, so $\angle QGA = 180^\circ - \angle T = \angle TPA + \angle TAP = 36^\circ + 56^\circ = 92^\circ$. Furthermore $GQ = PU = AG$, so $\triangle QGA$ is isosceles, and $\angle QAG = \frac{1}{2}(180^\circ - 92^\circ) = 44^\circ$. Because $\overline{MN}$ is a midline of $\triangle QPA$, it follows that $\overline{MN} \parallel \overline{AQ}$ and

$$\angle NMP = \angle QAP = \angle QAG + \angle GAP = 44^\circ + 56^\circ = 100^\circ,$$

so acute $\angle NMA = 80^\circ$. (Note that the value of the common length $PU = AG$ is immaterial.)



OR

Place the figure in the coordinate plane with $P = (-5, 0)$, $M = (0, 0)$, $A = (5, 0)$, and T in the first quadrant. Then

$$U = (-5 + \cos 36^\circ, \sin 36^\circ) \quad \text{and} \quad G = (5 - \cos 56^\circ, \sin 56^\circ),$$

and the midpoint N of $\overline{UG}$ is

$$\left(\frac{1}{2}(\cos 36^\circ - \cos 56^\circ), \frac{1}{2}(\sin 36^\circ + \sin 56^\circ)\right).$$

The tangent of $\angle NMA$ is the slope of line MN, which is calculated as follows using the sum-to-product trigonometric identites:

$$\begin{aligned}\tan(\angle NMA) &= \frac{\sin 36^\circ + \sin 56^\circ}{\cos 36^\circ - \cos 56^\circ} \\ &= \frac{2\sin\frac{36^\circ+56^\circ}{2}\cos\frac{36^\circ-56^\circ}{2}}{-2\sin\frac{36^\circ+56^\circ}{2}\sin\frac{36^\circ-56^\circ}{2}} \\ &= \frac{\cos 10^\circ}{\sin 10^\circ} = \cot 10^\circ = \tan 80^\circ,\end{aligned}$$

and it follows that $\angle NMA = 80^\circ$.

24. **Answer (B):** Because Alice and Bob are choosing their numbers uniformly at random, the cases in which two or three of the chosen numbers are equal have probability 0 and can be ignored. Suppose Carol chooses the number c. She will win if her number is greater than Alice's number and less than Bob's, and she will win if her number is less than Alice's number and greater than Bob's. There are three cases.

• If $c \le \frac{1}{2}$, then Carol's number is automatically less than Bob's, so her chance of winning is the probability that Alice's number is less than c, which is just c. The best that Carol can do in this case is to choose $c = \frac{1}{2}$, in which case her chance of winning is $\frac{1}{2}$.

• If $c \ge \frac{2}{3}$, then Carol's number is automatically greater than Bob's, so her chance of winning is the probability that Alice's number is greater than c, which is just $1 - c$. The best that Carol can do in this case is to choose $c = \frac{2}{3}$, in which case her chance of winning is $\frac{1}{3}$.

• Finally suppose that $\frac{1}{2} < c < \frac{2}{3}$. The probability that Carol's number is less than Bob's is

$$\frac{\frac{2}{3} - c}{\frac{2}{3} - \frac{1}{2}} = 4 - 6c,$$

so the probability that her number is greater than Alice's and less than Bob's is $c(4 - 6c)$. Similarly, the probability that her number is less than Alice's and greater than Bob's is $(1 - c)(6c - 3)$. Carol's probability of winning in this case is therefore

$$c(4 - 6c) + (1 - c)(6c - 3) = -12c^2 + 13c - 3.$$

The value of a quadratic polynomial with a negative coefficient on its quadratic term is maximized at $\frac{-b}{2a}$, where a is the coefficient on its quadratic term and b is the coefficient on its linear term; here that is when $c = \frac{13}{24}$, which is indeed between $\frac{1}{2}$ and $\frac{2}{3}$. Her probability of winning is then

$$-12 \cdot \left(\frac{13}{24}\right)^2 + 13 \cdot \frac{13}{24} - 3 = \frac{25}{48} > \frac{24}{48} = \frac{1}{2}.$$

Because the probability of winning in the third case exceeds the probabilities obtained in the first two cases, Carol should choose $\frac{13}{24}$.

25. **Answer (D):** The equation $C_n - B_n = A_n^2$ is equivalent to

$$c \cdot \frac{10^{2n} - 1}{9} - b \cdot \frac{10^n - 1}{9} = a^2 \left(\frac{10^n - 1}{9}\right)^2.$$

Dividing by $10^n - 1$ and clearing fractions yields

$$(9c - a^2) \cdot 10^n = 9b - 9c - a^2.$$

As this must hold for two different values n_1 and n_2, there are two such equations, and subtracting them gives

$$\left(9c - a^2\right)\left(10^{n_1} - 10^{n_2}\right) = 0.$$

The second factor is non-zero, so $9c - a^2 = 0$ and thus $9b - 9c - a^2 = 0$. From this it follows that $c = \left(\frac{a}{3}\right)^2$ and $b = 2c$. Hence digit a must be 3, 6, or 9, with corresponding values 1, 4, or 9 for c, and 2, 8, or 18 for b. The case $b = 18$ is invalid, so there are just two triples of possible values for a, b, and c, namely $(3, 2, 1)$ and $(6, 8, 4)$. In fact, in these cases, $C_n - B_n = A_n^2$ for *all* positive integers n; for example, $4444 - 88 = 4356 = 66^2$. The second triple has the greater coordinate sum, $6 + 8 + 4 = 18$.

Problems and solutions were contributed by Risto Atanasov, George Bauer, Chris Bolognese, Ivan Borsenco, Thomas Butts, Barbara Currier, Steven Davis, Steve Dunbar, Marta Eso, Zuming Feng, Zachary Franco, Peter Gilchrist, Jerrold Grossman, Chris Jeuell, Jonathan Kane, Joe Kennedy, Michael Khoury, Hugh Montgomery, Mohamed Omar, Albert Otto, Joachim Rebholz, David Wells, Barry Weng, and Carl Yerger.

The

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