The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

57th Annual American Mathematics Contest 12

AMC 12 – Contest B

Solutions Pamphlet

Wednesday, February 15, 2006

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules*

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1. **(C)** Because

$$(-1)^k = \begin{cases} 1, & \text{if } k \text{ is even,} \\ -1, & \text{if } k \text{ is odd,} \end{cases}$$

the sum can be written as

$$(-1+1)+(-1+1)+\cdots+(-1+1) = 0+0+\cdots+0 = 0.$$

2. **(A)** Because $4\spadesuit 5 = (4+5)(4-5) = -9$, it follows that

$$3\spadesuit(4\spadesuit 5) = 3\spadesuit(-9) = (3+(-9))(3-(-9)) = (-6)(12) = -72.$$

3. **(A)** Let c and p represent the number of points scored by the Cougars and the Panthers, respectively. The two teams scored a total of 34 points, so $c+p = 34$. The Cougars won by 14 points, so $c-p = 14$. The solution is $c = 24$ and $p = 10$, so the Panthers scored 10 points.

4. **(A)** The five items cost approximately $8+5+3+2+1 = 19$ dollars, so Mary's change is about \$1.00, which is 5 percent of her \$20.00.

5. **(A)** In order to catch up to John, Bob must walk 1 mile farther in the same amount of time. Because Bob's speed exceeds John's speed by $5-3 = 2$ miles per hour, the time required for Bob to catch up to John is $1/2$ hour, or 30 minutes.

6. **(B)** Francesca's 600 grams of lemonade contains $25+386 = 411$ calories, so 200 grams of her lemonade contains $411/3 = 137$ calories.

7. **(B)** There are only two possible occupants for the driver's seat. After the driver is chosen, any of the remaining three people can sit in the front, and there are two arrangements for the other two people in the back. Thus there are $2\cdot 3\cdot 2 = 12$ possible seating arrangements.

8. **(E)** Substituting $x = 1$ and $y = 2$ into the equations gives

$$1 = \frac{2}{4} + a \quad \text{and} \quad 2 = \frac{1}{4} + b.$$

It follows that

$$a + b = \left(1 - \frac{2}{4}\right) + \left(2 - \frac{1}{4}\right) = 3 - \frac{3}{4} = \frac{9}{4}.$$

OR

Because

$$a = x - \frac{y}{4} \quad \text{and} \quad b = y - \frac{x}{4}, \quad \text{we have} \quad a + b = \frac{3}{4}(x + y).$$

Since $x = 1$ when $y = 2$, this implies that $a + b = \frac{3}{4}(1 + 2) = \frac{9}{4}$.

9. **(B)** Let the integer have digits a, b, and c, read left to right. Because $1 \le a < b < c$, none of the digits can be zero and c cannot be 2. If $c = 4$, then a and b must each be chosen from the digits 1, 2, and 3. Therefore there are $\binom{3}{2} = 3$ choices for a and b, and for each choice there is one acceptable order. Similarly, for $c = 6$ and $c = 8$ there are, respectively, $\binom{5}{2} = 10$ and $\binom{7}{2} = 21$ choices for a and b. Thus there are altogether $3 + 10 + 21 = 34$ such integers.

10. **(A)** The sides of the triangle are x, $3x$, and 15 for some positive integer x. By the Triangle Inequality, these three numbers are the sides of a triangle if and only if $x + 3x > 15$ and $x + 15 > 3x$. Because x is an integer, the first inequality is equivalent to $x \ge 4$, and the second inequality is equivalent to $x \le 7$. Thus the greatest possible perimeter is $7 + 21 + 15 = 43$.

11. **(E)** Joe has 2 ounces of cream in his cup. JoAnn has drunk 2 ounces of the 14 ounces of coffee-cream mixture in her cup, so she has only $12/14 = 6/7$ of her 2 ounces of cream in her cup. Therefore the ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee is

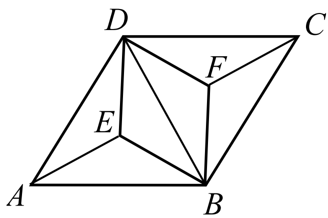
$$\frac{2}{\frac{6}{7} \cdot 2} = \frac{7}{6}.$$

12. **(D)** A parabola with the given equation and with vertex (p, p) must have equation $y = a(x - p)^2 + p$. Because the y-intercept is $(0, -p)$ and $p \ne 0$, it follows that $a = -2/p$. Thus

$$y = -\frac{2}{p}(x^2 - 2px + p^2) + p = -\frac{2}{p}x^2 + 4x - p,$$

so $b = 4$.

13. **(C)** Since $\angle BAD = 60°$, isosceles $\triangle BAD$ is also equilateral. As a consequence, $\triangle AEB$, $\triangle AED$, $\triangle BED$, $\triangle BFD$, $\triangle BFC$, and $\triangle CFD$ are congruent. These six triangles have equal areas and their union forms rhombus $ABCD$, so each has area $24/6 = 4$. Rhombus $BFDE$ is the union of $\triangle BED$ and $\triangle BFD$, so its area is 8.

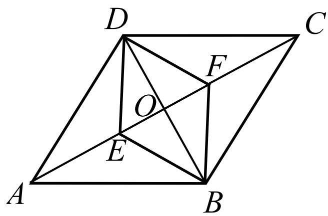


OR

Let the diagonals of rhombus $ABCD$ intersect at O. Since the diagonals of a rhombus intersect at right angles, $\triangle ABO$ is a $30-60-90°$ triangle. Therefore $AO = \sqrt{3} \cdot BO$. Because AO and BO are half the length of the longer diagonals of rhombi $ABCD$ and $BFDE$, respectively, it follows that

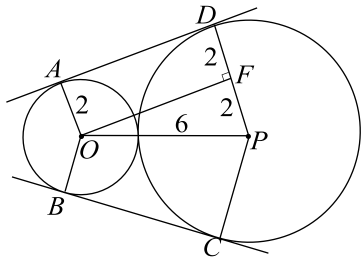
$$\frac{\text{Area}(BFDE)}{\text{Area}(ABCD)} = \left(\frac{BO}{AO}\right)^2 = \frac{1}{3}.$$

Thus the area of rhombus $BFDE$ is $(1/3)(24) = 8$.



14. **(D)** The total cost of the peanut butter and jam is $N(4B+5J) = 253$ cents, so N and $4B+5J$ are factors of $253 = 11 \cdot 23$. Because $N > 1$, the possible values of N are 11, 23, and 253. If $N = 253$, then $4B + 5J = 1$, which is impossible since B and J are positive integers. If $N = 23$, then $4B + 5J = 11$, which also has no solutions in positive integers. Hence $N = 11$ and $4B + 5J = 23$, which has the unique positive integer solution $B = 2$ and $J = 3$. So the cost of the jam is $11(3)(5¢) = \$1.65$.

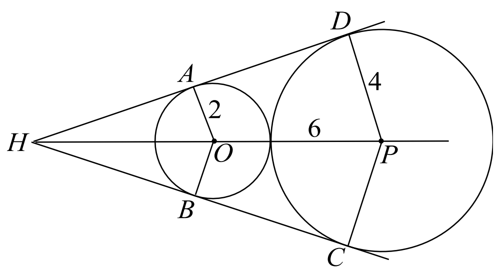
15. **(B)** Through O draw a line parallel to $\overline{AD}$ intersecting $\overline{PD}$ at F.



Then $AOFD$ is a rectangle and OPF is a right triangle. Thus $DF = 2$, $FP = 2$, and $OF = 4\sqrt{2}$. The area of trapezoid $AOPD$ is $12\sqrt{2}$, and the area of hexagon $AOBCPD$ is $2 \cdot 12\sqrt{2} = 24\sqrt{2}$.

OR

Lines $AD, BC,$ and OP intersect at a common point H.



Because $\angle PDH = \angle OAH = 90°$, triangles PDH and OAH are similar with ratio of similarity 2. Thus $2HO = HP = HO + OP = HO + 6$, so $HO = 6$ and $AH = \sqrt{HO^2 - OA^2} = 4\sqrt{2}$. Hence the area of $\triangle OAH$ is $(1/2)(2)(4\sqrt{2}) = 4\sqrt{2}$, and the area of $\triangle PDH$ is $(2^2)(4\sqrt{2}) = 16\sqrt{2}$. The area of the hexagon is twice the area of $\triangle PDH$ minus twice the area of $\triangle OAH$, so it is $24\sqrt{2}$.

16. **(C)** Diagonals $\overline{AC}$, $\overline{CE}$, $\overline{EA}$, $\overline{AD}$, $\overline{CF}$, and $\overline{EB}$ divide the hexagon into twelve congruent $30-60-90°$ triangles, six of which make up equilateral $\triangle ACE$. Because $AC = \sqrt{7^2 + 1^2} = \sqrt{50}$, the area of $\triangle ACE$ is $\frac{\sqrt{3}}{4}\left(\sqrt{50}\right)^2 = \frac{25}{2}\sqrt{3}$. The area of hexagon $ABCDEF$ is $2\left(\frac{25}{2}\sqrt{3}\right) = 25\sqrt{3}$.

OR

Let O be the center of the hexagon. Then triangles $ABC, CDE,$ and EFA are congruent to triangles $AOC, COE,$ and EOA, respectively. Thus the area of the hexagon is twice the area of equilateral $\triangle ACE$. Then proceed as in the first solution.

17. **(C)** On each die the probability of rolling k, for $1 \le k \le 6$, is

$$\frac{k}{1+2+3+4+5+6} = \frac{k}{21}.$$

There are six ways of rolling a total of 7 on the two dice, represented by the ordered pairs $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$. Thus the probability of rolling a total of 7 is

$$\frac{1\cdot 6 + 2\cdot 5 + 3\cdot 4 + 4\cdot 3 + 5\cdot 2 + 6\cdot 1}{21^2} = \frac{56}{21^2} = \frac{8}{63}.$$

18. **(B)** Each step changes either the x-coordinate or the y-coordinate of the object by 1. Thus if the object's final point is (a,b), then $a+b$ is even and $|a|+|b| \le 10$. Conversely, suppose that (a,b) is a lattice point with $|a|+|b| = 2k \le 10$. One ten-step path that ends at (a,b) begins with $|a|$ horizontal steps, to the right if $a \ge 0$ and to the left if $a < 0$. It continues with $|b|$ vertical steps, up if $b \ge 0$ and down if $b < 0$. It has then reached (a,b) in $2k$ steps, so it can finish with $5-k$ steps up and $5-k$ steps down. Thus the possible final points are the lattice points that have even coordinate sums and lie on or inside the square with vertices $(\pm 10, 0)$ and $(0, \pm 10)$. There are 11 such points on each of the 11 lines $x+y=2k$, $-5 \le k \le 5$, for a total of 121 different points.

19. **(B)** The 4-digit number on the license plate has the form *aabb* or *abab* or *baab*, where a and b are distinct integers from 0 to 9. Because Mr. Jones has a child of age 9, the number on the license plate is divisible by 9. Hence the sum of the digits, $2(a+b)$, is also divisible by 9. Because of the restriction on the digits a and b, this implies that $a+b=9$. Moreover, since Mr. Jones must have either a 4-year-old or an 8-year-old, the license plate number is divisible by 4. These conditions narrow the possibilities for the number to 1188, 2772, 3636, 5544, 6336, 7272, and 9900. The last two digits of 9900 could not yield Mr. Jones's age, and none of the others is divisible by 5, so he does not have a 5-year-old.

Note that 5544 is divisible by each of the other eight non-zero digits.

20. **(C)** The given condition is equivalent to $\lfloor \log_{10} x \rfloor = \lfloor \log_{10} 4x \rfloor$. Thus the condition holds if and only if

$$n \le \log_{10} x < \log_{10} 4x < n+1$$

for some negative integer n. Equivalently,

$$10^n \le x < 4x < 10^{n+1}.$$

This inequality is true if and only if

$$10^n \le x < \frac{10^{n+1}}{4}.$$

Hence in each interval $\left[10^n, 10^{n+1}\right)$, the given condition holds with probability

$$\frac{\left(10^{n+1}/4\right) - 10^n}{10^{n+1} - 10^n} = \frac{10^n((10/4) - 1)}{10^n(10 - 1)} = \frac{1}{6}.$$

Because each number in $(0, 1)$ belongs to a unique interval $\left[10^n, 10^{n+1}\right)$ and the probability is the same on each interval, the required probability is also $1/6$.

21. **(C)** Let $2a$ and $2b$, respectively, be the lengths of the major and minor axes of the ellipse, and let the dimensions of the rectangle be x and y. Then $x + y$ is the sum of the distances from the foci to point A on the ellipse, which is $2a$. The length of a diagonal of the rectangle is the distance between the foci of the ellipse, which is $2\sqrt{a^2 - b^2}$. Thus $x + y = 2a$ and $x^2 + y^2 = 4a^2 - 4b^2$. The area of the rectangle is

$$2006 = xy = \frac{1}{2}\left[(x + y)^2 - (x^2 + y^2)\right] = \frac{1}{2}\left[(2a)^2 - (4a^2 - 4b^2)\right] = 2b^2,$$

so $b = \sqrt{1003}$. Thus the area of the ellipse is

$$2006\pi = \pi ab = \pi a\sqrt{1003},$$

so $a = 2\sqrt{1003}$, and the perimeter of the rectangle is $2(x + y) = 4a = 8\sqrt{1003}$.

22. **(B)** Note that n is the number of factors of 5 in the product $a!b!c!$, and $2006 < 5^5$. Thus

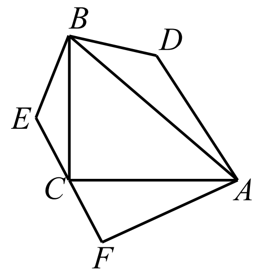
$$n = \sum_{k=1}^{4} \left(\lfloor a/5^k \rfloor + \lfloor b/5^k \rfloor + \lfloor c/5^k \rfloor\right).$$

Because $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor z \rfloor \geq \lfloor x + y + z \rfloor - 2$ for all real numbers x, y, and z, it follows that

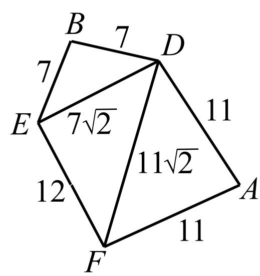
$$\begin{aligned} n &\geq \sum_{k=1}^{4} \left(\lfloor (a + b + c)/5^k \rfloor - 2\right) \\ &= \sum_{k=1}^{4} \left(\lfloor 2006/5^k \rfloor - 2\right) \\ &= 401 + 80 + 16 + 3 - 4 \cdot 2 = 492. \end{aligned}$$

The minimum value of 492 is achieved, for example, when $a = b = 624$ and $c = 758$.

23. **(E)** Let D, E, and F be the reflections of P about $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Then $\angle FAD = \angle DBE = 90°$, and $\angle ECF = 180°$. Thus the area of pentagon $ADBEF$ is twice that of $\triangle ABC$, so it is s^2.



Observe that $DE = 7\sqrt{2}$, $EF = 12$, and $FD = 11\sqrt{2}$. Furthermore, $(7\sqrt{2})^2 + 12^2 = 98 + 144 = 242 = (11\sqrt{2})^2$, so $\triangle DEF$ is a right triangle. Thus the pentagon can be tiled with three right triangles, two of which are isosceles, as shown.



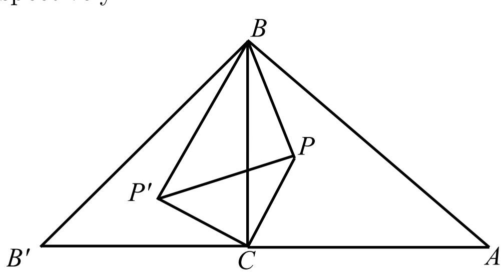
It follows that

$$s^2 = \frac{1}{2} \cdot (7^2 + 11^2) + \frac{1}{2} \cdot 12 \cdot 7\sqrt{2} = 85 + 42\sqrt{2},$$

so $a + b = 127$.

OR

Rotate $\triangle ABC$ 90° counterclockwise about C, and let B' and P' be the images of B and P, respectively.



Then $CP' = CP = 6$, and $\angle PCP' = 90^\circ$, so $\triangle PCP'$ is an isosceles right triangle. Thus $PP' = 6\sqrt{2}$, and $BP' = AP = 11$. Because $\left(6\sqrt{2}\right)^2 + 7^2 = 11^2$, the converse of the Pythagorean Theorem implies that $\angle BPP' = 90^\circ$. Hence $\angle BPC = 135^\circ$. Applying the Law of Cosines in $\triangle BPC$ gives

$$BC^2 = 6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cos 135^\circ = 85 + 42\sqrt{2},$$

and $a+b=127$.

24. **(C)** For a fixed value of y, the values of $\sin x$ for which $\sin^2 x - \sin x \sin y + \sin^2 y = \frac{3}{4}$ can be determined by the quadratic formula. Namely,

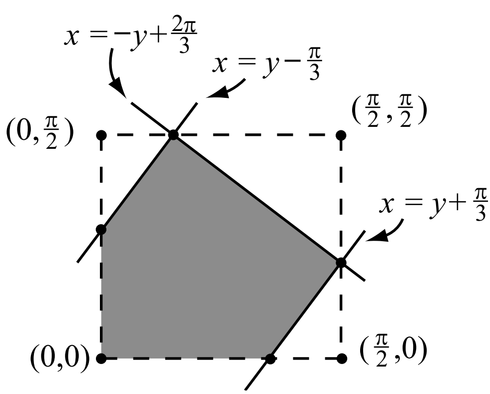
$$\sin x = \frac{\sin y \pm \sqrt{\sin^2 y - 4(\sin^2 y - \frac{3}{4})}}{2} = \frac{1}{2}\sin y \pm \frac{\sqrt{3}}{2}\cos y.$$

Because $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, this implies that

$$\sin x = \cos\left(\frac{\pi}{3}\right)\sin y \pm \sin\left(\frac{\pi}{3}\right)\cos y = \sin\left(y \pm \frac{\pi}{3}\right).$$

Within S, $\sin x = \sin(y - \frac{\pi}{3})$ implies $x = y - \frac{\pi}{3}$. However, the case $\sin x = \sin(y + \frac{\pi}{3})$ implies $x = y + \frac{\pi}{3}$ when $y \le \frac{\pi}{6}$, and $x = -y + \frac{2\pi}{3}$ when $y \ge \frac{\pi}{6}$. Those three lines divide the region S into four subregions, within each of which the truth value of the inequality is constant. Testing the points $(0,0)$, $(\frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2})$, and $(\frac{\pi}{2}, \frac{\pi}{2})$ shows that the inequality is true only in the shaded subregion. The area of this subregion is

$$\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\cdot\left(\frac{\pi}{3}\right)^2 - 2\cdot\frac{1}{2}\cdot\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{6}.$$



25. **(B)** The condition $a_{n+2} = |a_{n+1} - a_n|$ implies that a_n and a_{n+3} have the same parity for all $n \ge 1$. Because a_{2006} is odd, a_2 is also odd. Because $a_{2006} = 1$ and a_n is a multiple of $\gcd(a_1, a_2)$ for all n, it follows that $1 = \gcd(a_1, a_2) = \gcd(3^3 \cdot 37, a_2)$. There are 499 odd integers in the interval $[1, 998]$, of which 166 are multiples of 3, 13 are multiples of 37, and 4 are multiples of $3 \cdot 37 = 111$. By the Inclusion-Exclusion Principle, the number of possible values of a_2 cannot exceed $499 - 166 - 13 + 4 = 324$.

To see that there are actually 324 possibilities, note that for $n \ge 3$, $a_n < \max(a_{n-2}, a_{n-1})$ whenever a_{n-2} and a_{n-1} are both positive. Thus $a_N = 0$ for some $N \le 1999$. If $\gcd(a_1, a_2) = 1$, then $a_{N-2} = a_{N-1} = 1$, and for $n > N$ the sequence cycles through the values 1, 1, 0. If in addition a_2 is odd, then a_{3k+2} is odd for $k \ge 1$, so $a_{2006} = 1$.