2020 INTERNATIONAL MATH CONTEST

Practice Problems for Russian Math Olympiad

Grades 7-8

2019 INTERNATIONAL MATH CONTEST

ID ______________________
Name ______________________
Grade ______________________
RSM Affiliation ______________________
Test Location ______________________

FINAL ROUND

Grades 7-8

RSM FOUNDATION

1

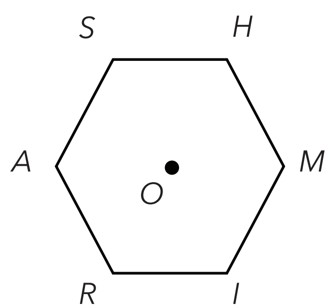
Given a 2019-sided red regular polygonal shape with side length 1, if each side also forms the side of a blue square shape located outside the red shape, what is the perimeter of the resulting red-and-blue shape?

2

Gary, Mary, and Rory have the same number of candies. If Gary gives Mary half of all his candies, then Mary gives Rory half of all the candies she has at the moment, and then Rory gives Gary half of all the candies he (Rory) has at the moment, Gary would have 12 more candies than he had originally. How many candies do Gary, Mary, and Rory have altogether?

3

The area of a regular hexagon *RASHMI* is 2019 square feet. Compute the area (in square feet) of the quadrilateral *RSMO* where *O* is the center of the hexagon.



4

Numbers were written in 1000 boxes in a row, one number per box (only the first ten and the last five boxes are shown). For every four boxes in a row, the sum of their numbers was 12. Most of the numbers got erased over time, but three of them remain. What number was written in the last box on the right?

	0					1				…		2			

5

If $\overline{RS} + \overline{SM} + \overline{MR} + X = 201$, compute $\overline{SR} + \overline{MS} + \overline{RM} + 7 \cdot X$.
(R, S, and M represent the digits of the 2-digit numbers $\overline{RS}$, $\overline{SM}$, $\overline{MR}$, $\overline{SR}$, $\overline{MS}$, and $\overline{RM}$; X also represents a digit.)

6

A race car moved 1 second at a constant rate of 68 m/sec, then 1 second at a constant rate of 69 m/sec, then 1 second at a constant rate of 70 m/sec, and so on. All movements were in the same direction. In how many seconds would the total distance covered by the race car be 2 kilometers?

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FINAL ROUND

Grades 7-8

RSM FOUNDATION

7

How many different positive integers are there containing only the digits 1, 2, and/or 3 (each of these digits can be used one or more times or not at all) such that for each of these integers, the sum of all of its digits equals seven?

8

Five friends are all of different heights. The average height of the three tallest friends is exactly the height of one of them. The average height of the four tallest friends is exactly the height of one of them. The average height of all five friends is exactly the height of one of them. The second-tallest friend is 16 cm taller than the second-shortest one. The tallest friend is taller than the shortest one by how many centimeters?

9

Find the positive integer value of x if $\dfrac{1}{a+\dfrac{1}{a+\dfrac{1}{x}}} = \dfrac{19}{63}$ where a is a positive integer.

10

In a triangle RSM, the measure of angle SRM is twice the measure of angle RSM. A point O is selected on side $\overline{RS}$ such that $SO = SM$. The length of the angle bisector of angle RMS equals RO. What is the degree measure of angle RSM?

11

The number R has exactly 7 different positive integer factors, the number S has exactly 8 different positive integer factors, and their product $R \cdot S$ has exactly M different positive integer factors. Compute the sum of all different possible values of M.

12

Say that a positive integer is "five-important" if it is a multiple of 5 and/or contains the digit 5. For instance, the numbers 55, 120, and 456 are five-important, but the number 2019 is not. Say that a number is "super-five-important" if it is five-important and remains five-important after erasing any one of its digits. For instance, the numbers 5070, 5005, and 5577 are super-five-important, but the numbers 5, 100, 2019, and 2015 are not. How many different super-five-important numbers are there between 1 and 2019?

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2019 International Math Contest

RSM Foundation

Final Round Answers

Problem/Grades	7-8
1	6057
2	96
3	673
4	9
5	219
6	25
7	44
8	68
9	6
10	20
11	122
12	116

2018
INTERNATIONAL MATH CONTEST

ID ______________________________

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FINAL ROUND

Grades 7-8

RSM FOUNDATION

1 Find the fifth-smallest seven-digit multiple of 2018.

2 A bag contains 18 small red, 19 small black, 13 large black, and 15 large red T-shirts, and nothing else. What is the least number of T-shirts Alpa must take out of the bag (without looking at them) to be absolutely sure of having at least two T-shirts among them that differ only by size or only by color?

3 Several RSM students solved a total of 101 different problems. Each of these problems was solved by exactly one student, and each student solved a different number of problems. The number of problems each student solved was a prime. What is the greatest possible number of these students?

4 A rectangle is rotated 90° around its center. Both rectangular shapes (the original and the new ones) define a square as their intersection and a 12-gon as their union. The perimeter of the 12-gon is 100 times the perimeter of the square. The area of the 12-gon is X times the area of the square. Compute the value of X.

5 Three apples weigh as much as five kiwis. Five apples weigh as much as eight mandarins. What is the least possible odd total number of kiwis and mandarins that could be split into two groups of the same weight, without cutting any of the kiwis and/or mandarins? Assume that all apples weigh the same, all kiwis weigh the same, and all mandarins weigh the same.

6 The average degree measure of an interior angle of an N-gon is 140°. Compute the average degree measure of an exterior angle of an $(N + 1)$-gon. (An exterior angle of a polygon is an angle formed outside the polygon by one of its sides and an extension of an adjacent side.)

7 The product of a three-digit number and all of its non-zero digits equals X^Y, where X and Y are positive integers. Compute the greatest possible value of Y.

Please fold over on line. Write answers on back.

FINAL ROUND

Grades 7-8

2018 INTERNATIONAL MATH CONTEST

RSM FOUNDATION

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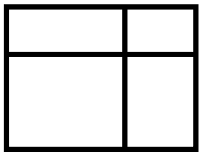
8

A rectangular shape is divided into four non-overlapping rectangular shapes as shown in the diagram. The areas of these four regions (in some order) are 27 square feet, 20 square feet, 18 square feet, and N square feet, where N is an integer. Compute the area (in square feet) of the original rectangle.



9

The numerator and the denominator of a fraction are positive integers. If both of them are increased by 1, the value of the new fraction would be 0.1 greater than the value of the original fraction. Compute the number of different original fractions satisfying the above conditions.

10

In Numberland there is a cube, and all of the numbers live on its edges (some live on the corners). For each of the 12 edges, Olga calculated the sum of all of the numbers living on that edge (including the corners), and she got (in some order) all different positive integers from 1 through 12. The sum of all of the numbers in Numberland equals 50. Find the sum of all of the numbers living on the corners of the cube.

11

Say that a number is "striped" if all its digits are different, and digits of the same parity (even or odd) do not appear next to each other. For example, the numbers 1 and 218 are striped, but the numbers 900 and 2018 are not. How many different striped ten-digit numbers are there?

12

Say that a positive integer K is "boring" if none of the factorials ends in exactly K zeroes. For example, the number 1 is not boring since $6! = 1\times2\times3\times4\times5\times6 = 720$ ends in exactly one zero. Say that a positive integer K is "not-so-boring" if all three numbers $K - 1$, K, and $K + 1$ are boring. Find the smallest not-so-boring number. (Note that N factorial, written as "$N!$", is the product of all integers from 1 through N, where N is a positive integer.)

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2018 International Math Contest

RSM Foundation

Final Round Answers

Problem/Grades	7-8
1	1009000
2	35
3	8
4	199
5	49
6	36
7	11
8	95
9	2
10	14
11	25920
12	154

2017
INTERNATIONAL MATH CONTEST

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FINAL ROUND

Grades 7-8

RSM FOUNDATION

1

The sum of all three pairwise products of the numbers R, S, and M is 99 less than the sum of all three pairwise products of the numbers $R + 1$, $S + 1$, and $M + 1$. Compute $R + S + M$.

2

The sum of five natural numbers is 210. Not all of them have the same value. Find the least possible value of the largest of these numbers.

3

A teacher asked her students to find a 3-digit positive integer with the product of all its digits equal to 128. Jen realized that there was more than one such number, and listed each of them once. Find the sum of all of Jen's numbers.

4

In a triangle with perimeter 2017, all sides have integer lengths (in feet). One side is 10 feet shorter than another one. One side is 20 feet longer than another one. Compute the length (in feet) of the medium side of the triangle.

5

Anna and Oleg are collecting natural numbers. Anna is collecting only numbers with different digits (such as 2017), and so far she has collected all such numbers up to 1023. Oleg is collecting only prime numbers, and so far he has collected all such numbers up to 2017. What is the largest number which appears in both Anna's and Oleg's collections now?

6

A metal letter R weighs 2 pounds, a metal letter S weighs 1 pound, and a metal letter M weighs 4 pounds. If you took a certain 10-letter "word" containing only the metal letters R, S, and M (at least one of each) and simultaneously replaced all letters R with S, all letters S with M, and all letters M with R, the total weight of all letters in this word would not change. Compute this total weight (in pounds).

Please fold over on line. Write answers on back.

2017
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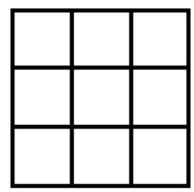
FINAL ROUND

Grades 7-8

RSM FOUNDATION

7 A horse ran at a constant speed and covered 20 km and 17 m in 20 min and 17 sec. The next day she ran at the same speed and covered 20 km and X m in 20 min and 30 sec. Compute the value of X rounded to the nearest integer.

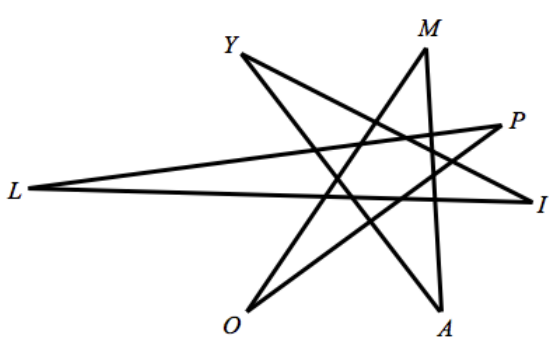
8 How many different ways are there to place nine different digits from 1 to 9 inside the nine square cells of a 3-by-3 grid (one digit per cell) such that for every pair of consecutive digits their square cells share a side?



9 All possible diagonals drawn from the two adjacent vertices A and B of a regular hectogon divide the hectogon's interior into a number of non-overlapping shapes – triangles and quadrilaterals (without any part of a line inside them). How many of these shapes are quadrilaterals? (A hectogon is a polygon with 100 sides.)

10 Serena took two numbers which may or may not be integers, rounded each of them up to the nearest integer, multiplied the results, and got 100. When she took the original numbers, rounded each of them down to the nearest integer and multiplied the results, she got X. Find the largest possible value of X.

11 Let $D°$ be the total degree measure of the seven internal angles of an irregular heptagonal star whose vertices are O, L, Y, M, P, I, and A (see the diagram). Compute the value of D.



12 We define an RSM-word as a 6-letter word containing two letters R, two letters S, and two letters M, in which there is at least one way that Rosemary can circle three letters such that the circled letters read (from left to right) R-S-M. How many different RSM-words are there?

Please fold over on line. Write answers on back.

Answers:

Problem No.	Answer
1	48
2	43
3	3774
4	669
5	983
6	21
7	231
8	40
9	4656
10	202
11	180
12	47

2016
INTERNATIONAL
MATH CONTEST

First Name ______________________
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FINAL ROUND
Grades 7-8

RSM FOUNDATION

1

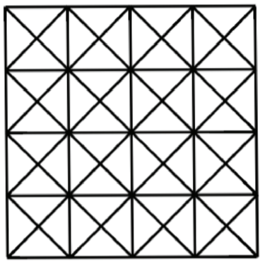
After multiplying $r + 20s$, $r + 16m$, and $s + 2016m$ and simplifying the resulting expression, what would be the coefficient of the rsm term? (For example, in $20x + 16xy + 2016xyz$, 2016 is the coefficient of the xyz term.)

2

There are three cars in the RSM garage: a Rolls Royce, a Studebaker, and a Maserati. One of them is red, one of them is silver, and one of them is magenta. The silver car is $2016 more expensive than the Studebaker. The magenta car is $6102 less expensive than the Maserati. By how many dollars is the red car more expensive than the Rolls Royce?

3

The diagram consists of a 4-by-4 square divided into 16 unit squares, and all the diagonals of these 16 unit squares. How many squares of all sizes and positions are there in this diagram, including squares that are made up of other squares?



4

Alice thought of a two-digit number. If she reverses the order of the digits, the new number would be twice as large as the original number increased by 1. What was the number Alice thought of?

5

Irina took the word GEOMETRY and replaced each of its eight letters with a number according to the code A = 1, B = 2, …, Z = 26. Then she multiplied six of these numbers to get a cube of an integer. Find this integer.

6

Venus took 30 numbers, rounded each of them up to the nearest integer, added the results, and got 2016. When she took the original numbers, rounded each of them down to the nearest integer and added the results, she got 2000. How many of her original numbers were integers?

Please fold over on line. Write answers on back.

2016 INTERNATIONAL MATH CONTEST

First Name ______________________
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FINAL ROUND

Grades 7-8

RSM FOUNDATION

7 The President of Dollarstan is deciding between two income tax plans. According to one of the plans, all residents would pay tax equal to 10% of their yearly income (if this income is positive). According to the other plan, the first 150,000 D-dollars of a resident's yearly income would not be taxed, and the tax (if any) would equal 16% of any yearly income over 150,000 D-dollars. The President cannot decide which tax plan to propose because his own tax under either plan is the same. What is the yearly income (in D-dollars) of the President of Dollarstan? Note that this income is a positive number.

8 On Monday Ravi wrote a huge positive integer on the board. On Tuesday he wrote the digit 2 near every odd digit on the board. On Wednesday he wrote the digit 3 near every even digit on the board. At this point the total number of digits on the board was 2016 more than the total number of digits in Ravi's original huge number (from Monday). What was the total number of digits on the board at the end of Tuesday?

9 A square is drawn on each side of a triangle (each side of the triangle is a side of one of the squares). None of the four shapes overlap. The ratio of the areas of the squares is 1:2:3. What is the degree measure of the largest angle of the triangle?

10 Rick, Sol, and Mike are in the same RSM class and have the same height. The average height of all students in the class except Rick is 63.1 inches. The average height of all students in the class except Rick and Sol is 62 inches. The average height of all students in the class except Rick, Sol, and Mike is 60.625 inches. How many students are in this class?

11 All 6-digit positive integers containing each of the digits 1, 2, 3, 4, 5, 6 exactly once are written down on the board. How many of these numbers are divisible by 8?

12 Anton took three sides of an equilateral triangle with perimeter 2016, cut one of the sides into two equal parts, and made a quadrilateral which is not a parallelogram using these four segments. What is the greatest possible area of this quadrilateral?

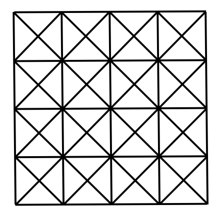
Please fold over on line. Write answers on back.

2016 RSM Olympiad 7-8

1. After multiplying $r + 20s$, $r + 16m$, and $s + 2016m$ and simplifying the resulting expression, what would be the coefficient of the rsm term? (For example, in $20x + 16xy + 2016xyz$, 2016 is the coefficient of the xyz term.)
Answer: 40336
Solution. After multiplying we get two rsm terms: $r\times16m\times s = 16rsm$ and $20s\times r\times2016m = 40320rsm$. After simplifying we get one rsm term $(16 + 40320)rsm = 40336rsm$, so the answer is 40,336 (or 40336).

2. There are three cars in the RSM garage: a Rolls Royce, a Studebaker, and a Maserati. One of them is red, one of them is silver, and one of them is magenta. The silver car is \$2016 more expensive than the Studebaker. The magenta car is \$6102 less expensive than the Maserati. By how many dollars is the red car more expensive than the Rolls Royce?
Answer: 4086
Solution. Let the Rolls Royce cost R dollars, the Studebaker cost S dollars, and the Maserati cost M dollars. Then the total cost of all three cars equals $R + S + M$ dollars. The silver car costs $S + 2016$ dollars, and the magenta car costs $M – 6102$ dollars, so their total cost equals $(S + 2016) + (M – 6102) = S + M – 4086$ dollars. But together with the red car they cost $R + S + M$ dollars, therefore the cost of the red car equals $(R + S + M) – (S + M – 4086) = R + 4086$ dollars. Since R is the cost of the Rolls Royce, the red car is \$4086 more expensive than the Rolls Royce.

3. The diagram consists of a 4-by-4 square divided into 16 unit squares, and all the diagonals of these 16 unit squares. How many squares of all sizes and positions are there in this diagram, including squares that are made up of other squares?



Answer: 72
Solution. Let's count squares by their types and sizes. Square sides could be parallel either to the sides or to the diagonals of the original 4×4 square. First, let's consider squares with sides parallel to the sides of the original 4×4 square. The diagram contains only 1×1, 2×2, 3×3, and 4×4 such squares. There are 16 1×1 squares, 9 2×2 squares, 4 3×3 squares, and 1 4×4 square in the diagram, for a total of 16 + 9 + 4 + 1 = 30 squares.
Now let's consider squares with sides parallel to the diagonals of the original 4×4 square (we call them "diagonal" squares). Call a triangle in the diagram small if it is not made up of other triangles. The diagram contains only the following types of diagonal squares: small (made up of 2 small triangles), medium (made up of 4 small diagonal squares), large (made up of 9 small diagonal squares), and extra-large (made up of 16 small diagonal squares). There are 24 small, 13 medium, 4 large, and 1 extra-large diagonal squares in the diagram, for a total of 24 + 13 + 4 + 1 = 42 diagonal squares. Thus, there are 30 + 42 = 72 squares of all sizes and positions in the diagram.

4. Alice thought of a two-digit number. If she reverses the order of the digits, the new number would be twice as large as the original number increased by 1. What was the number Alice thought of?
Answer: 25
Solution. Let Alice's two-digit number be $\underline{TU}$, where T is the tens digit and U is the units digit. It could also be written as $10T + U$. If Alice reverses the order of the digits, the new number will be $\underline{UT} = 10U + T$. Since the new number is twice as large as the original

number increased by 1, we obtain the following equation: $10U + T = 2(10T + U + 1)$, which could be simplified to $8U – 2 = 19T$. Since $8U$ and 2 are even, $19T$ must be even, and therefore T must be an even digit. Note that $T > 0$ (Alice thought of a two-digit number). If $T = 2$, $19T + 2 = 40$, so $U = 5$. If $T \geq 4$, $19T \geq 76$ which is greater than $8U – 2$ (the largest possible value of $8U – 2$ is $8\times9 – 2 = 70$). Since $T = 2$ and $U = 5$, the number Alice thought of was 25.

5. Irina took the word GEOMETRY and replaced each of its eight letters with a number according to the code A = 1, B = 2, …, Z = 26. Then she multiplied six of these numbers to get a cube of an integer. Find this integer.
Answer: 150
Solution. First, let's recall the order of the letters in the alphabet to get the numbers Irina used to replace letters: $G = 7$, $E = 5$, $O = 15$, $M = 13$, $E = 5$, $T = 20$, $R = 18$, $Y = 25$. Note that 7 and 13 are primes, and none of the other six numbers is a multiple of either 7 or 13. Irina had to exclude numbers 7 and 13 from multiplication, otherwise the result would be a multiple of 7 but not a multiple of 7^3 (or a multiple of 13 but not a multiple of 13^3), and therefore the result would not be a cube of an integer. Thus, Irina multiplied numbers 5, 15, 5, 20, 18, and 25, so she got $5\times15\times5\times20\times18\times25 = 5\times(3\times5)\times5\times(2\times2\times5)\times(2\times3\times3)\times(5\times5) = (2\times3\times5\times5)^3 = 150^3$. Note that there is only one integer whose cube is 150^3, namely 150, so the answer is 150.

6. Venus took 30 numbers, rounded each of them up to the nearest integer, added the results, and got 2016. When she took the original numbers, rounded each of them down to the nearest integer and added the results, she got 2000. How many of her original numbers were integers?
Answer: 14
Solution. If any of Venus's 30 original numbers (let it be x) was an integer, then its value did not change when rounding up or down to the nearest integer. (Formally, $\lceil x \rceil = \lfloor x \rfloor = x$ for any integer x, where $\lceil y \rceil$ is the result of rounding a number y up to the nearest integer, and $\lfloor y \rfloor$ is the result of rounding a number y down to the nearest integer.) Therefore $\lceil x \rceil$ contributed to the first sum as much as $\lfloor x \rfloor$ contributed to the second sum. If any of Venus's 30 original numbers (let it be x) was not an integer, then its value increased (by less than 1) when rounding up to the nearest integer, and decreased (by less than 1) when rounding down to the nearest integer. The results of these two rounding operations are consecutive integers. (Formally, $\lfloor x \rfloor < x < \lceil x \rceil$ and $\lceil x \rceil – \lfloor x \rfloor = 1$ for any non-integer x.) Therefore $\lceil x \rceil$ contributed to the first sum 1 more than $\lfloor x \rfloor$ contributed to the second sum. Thus, the value of the first sum (2016) is greater than the value of the second sum (2000) by the total number of non-integers among Venus's 30 original numbers. So, there were 2016 – 2000 = 16 non-integers and 30 – 16 = 14 integers among these 30 numbers.

7. The President of Dollarstan is deciding between two income tax plans. According to one of the plans, all residents would pay tax equal to 10% of their yearly income (if this income is positive). According to the other plan, the first 150,000 D-dollars of a resident's yearly income would not be taxed, and the tax (if any) would equal 16% of any yearly income over 150,000 D-dollars. The President cannot decide which tax plan to propose because his own tax under either plan is the same. What is the yearly income (in D-dollars) of the President of Dollarstan? Note that this income is a positive number.
Answer: 400,000 (or 400000)

Solution. Let $x > 0$ be the yearly income (in D-dollars) of the President of Dollarstan. Since $x > 0$, under the first plan the President would pay tax equal to 10% of his yearly income, i.e. $0.1x > 0$. If $x \leq 150000$, then under the second plan the President would pay no tax, which is different from $0.1x$. Therefore $x > 150000$, and under the second plan the first 150000 D-dollars of the President's yearly income would not be taxed, but the President would still pay tax equal to 16% of his yearly income over 150000 D-dollars, i.e. $0.16(x - 150000)$. Since the President's tax under either plan is the same, we get the following equation: $0.1x = 0.16(x - 150000)$. Simplifying, we get $0.06x = 24000 \Leftrightarrow x = 400000$ which indeed greater than 150000. Thus, the yearly income (in D-dollars) of the President of Dollarstan is 400000.

Note. The above solution assumes that the President of Dollarstan is a resident of Dollarstan, or at least pays income tax as a resident.

8. On Monday Ravi wrote a huge positive integer on the board. On Tuesday he wrote the digit 2 near every odd digit on the board. On Wednesday he wrote the digit 3 near every even digit on the board. At this point the total number of digits on the board was 2016 more than the total number of digits in Ravi's original huge number (from Monday). What was the total number of digits on the board at the end of Tuesday?
Answer: 2016
Solution. Let x be the number of odd digits and y be the number of even digits in Ravi's original huge number. There was a total of $x + y$ digits on the board at the end of Monday. On Tuesday Ravi wrote the digit 2 on the board x times. Since 2 is even, there were x odd and $y + x$ even digits on the board at the end of Tuesday, for a total of $2x + y$ digits. On Wednesday Ravi wrote the digit 3 on the board $y + x$ times. Since 3 is odd, there were $x + (y + x) = 2x + y$ odd and $y + x$ even digits on the board at the end of Wednesday, for a total of $3x + 2y$ digits. Since this number was 2016 more than the total number of digits in Ravi's original huge number (from Monday), we get the following equation: $(3x + 2y) - (x + y) = 2016$ which is equivalent to $2x + y = 2016$. But $2x + y$ was precisely the total number of digits on the board at the end of Tuesday, so the answer is 2016.

9. A square is drawn on each side of a triangle (each side of the triangle is a side of one of the squares). None of the four shapes overlap. The ratio of the areas of the squares is 1:2:3. What is the degree measure of the largest angle of the triangle?
Answer: 90
Solution. Let a, b, c be the side lengths of the triangle, $a \leq b \leq c$. Then the areas of the squares described in the problem statement would be $a^2 \leq b^2 \leq c^2$. Since the ratio of the areas of the squares is 1:2:3, we get $a^2:b^2:c^2 = 1:2:3$, or $b^2 = 2a^2$, $c^2 = 3a^2$. Therefore $a^2 + b^2 = a^2 + 2a^2 = 3a^2 = c^2$, so the triangle is a right one by the converse of the Pythagorean Theorem. Since a right triangle contains one 90° angle and two acute angles (each measuring less than 90°), the largest angle of the triangle measures 90°.

10. Rick, Sol, and Mike are in the same RSM class and have the same height. The average height of all students in the class except Rick is 63.1 inches. The average height of all students in the class except Rick and Sol is 62 inches. The average height of all students in the class except Rick, Sol, and Mike is 60.625 inches. How many students are in this class?
Answer: 11

Solution 1. Since $63.1 = 63\frac{1}{10}$ and $60.625 = 60\frac{5}{8}$, one possibility is the following: there are 10 students in the class without Rick, and 8 (precisely 2 less than 10) students in the class without Rick, Sol, and Mike. In this case the total height of all students in the class except Rick, Sol, and Mike is $8 \times 60\frac{5}{8} = 485$ inches. The total height of all students in the class except Rick and Sol is $9 \times 62 = 558$ inches, therefore Mike is 558 – 485 = 73 inches tall. The total height of all students in the class except Rick is $10 \times 63\frac{1}{10} = 631$ inches, therefore Sol is 631 – 558 = 73 inches tall as well. Since this possibility satisfies all the conditions of the problem (there are no more restrictions on Rick's height except that Rick, Sol, and Mike have the same height), the answer is that there are 10 + 1 = 8 + 3 = 11 students in the class.

Solution 2. Let $N > 3$ be the number of students in the class, let Rick, Sol, and Mike each be x inches tall, and let the total height of all other $N – 3$ students be T inches. Then the total height of all students in the class except Rick and Sol (there are $N – 2$ of them) is $T + x$ inches, and the total height of all students in the class except Rick (there are $N – 1$ of them) is $T + 2x$ inches. So, the conditions of the problem could be written as the following equations: $\frac{T+2x}{N-1} = 63.1$, $\frac{T+x}{N-2} = 62$, and $\frac{T}{N-3} = 60.625$. Let's re-write the equations in equivalent form: $T + 2x = 63.1(N – 1)$, $T + x = 62(N – 2)$, $T = 60.625(N – 3)$. Subtracting the first two equations from each other yields $x = (63.1 – 62)N + (124 – 63.1) = 1.1N + 60.9$, and subtracting the last two equations from each other yields $x = (62 – 60.625)N + (181.875 – 124) = 1.375N + 57.875$. Thus, $1.1N + 60.9 = 1.375N + 57.875 \Leftrightarrow 0.275N = 3.025 \Leftrightarrow 275N = 3025 \Leftrightarrow N = 11$. We can now find x and T to verify that $N = 11$ is not an extraneous solution.

11. All 6-digit positive integers containing each of the digits 1, 2, 3, 4, 5, 6 exactly once are written down on the board. How many of these numbers are divisible by 8?
Answer: 84
Solution. Recall the following divisibility rules. An integer is divisible by 2 if and only if its units digit is even (divisible by 2). An integer is divisible by 4 if and only if its tens and units digits read in that order as a 2-digit number (leading 0s are allowed) is divisible by 4. An integer is divisible by 8 if and only if its hundreds, tens, and units digits read in that order as a 3-digit number (leading 0s are allowed) is divisible by 8. Also, for an integer to be divisible by 8, it must be divisible by 2 and by 4 (although divisibility by 2 and by 4 does not guarantee divisibility by 8).

Now let's consider all 3-digit positive integers containing only the digits 1, 2, 3, 4, 5, 6, and not containing duplicate digits. Let's count how many of these numbers are divisible by 8. For a 3-digit number $\underline{abc}$ to be divisible by 8, its units digit c must be even, i.e. it must be 2, 4, or 6.

Case 1: $c = 2$. For a 3-digit number $\underline{ab2}$ to be divisible by 8, 2-digit number $\underline{b2}$ must be divisible by 4. Since 2 is even but not divisible by 4, 2-digit number $\underline{b0}$ must also be even (which is always true) and not divisible by 4, so b must be odd (1, 3, or 5).

Sub-case 1.1: $b = 1$. Consider number $\underline{a12}$. We want it to be divisible by 8. Since 12 is divisible by 4 but not divisible by 8, 3-digit number $\underline{a00}$ must also be divisible by 4 (which is always true) and not divisible by 8, so a must be odd (3 or 5).

Sub-case 1.2: $b = 3$. Consider number $\underline{a32}$. We want it to be divisible by 8. Since 32 is divisible by 8, 3-digit number $\underline{a00}$ must also be divisible by 8, so a must be even (4 or 6).

Sub-case 1.3: $b = 5$. Consider number $\underline{a52}$. We want it to be divisible by 8. Since 52 is divisible by 4 but not divisible by 8, 3-digit number $\underline{a00}$ must also be divisible by 4 (which is always true) and not divisible by 8, so a must be odd (1 or 3).

Case 2: $c = 4$. For a 3-digit number $\underline{ab4}$ to be divisible by 8, 2-digit number $\underline{b4}$ must be divisible by 4. Since 4 is divisible by 4, 2-digit number $\underline{b0}$ must also be divisible by 4, so b must be even (2 or 6).

Sub-case 2.1: $b = 2$. Consider number $\underline{a24}$. We want it to be divisible by 8. Since 24 is divisible by 8, 3-digit number $\underline{a00}$ must also be divisible by 8, so a must be even (6).

Sub-case 2.2: $b = 6$. Consider number $\underline{a64}$. We want it to be divisible by 8. Since 64 is divisible by 8, 3-digit number $\underline{a00}$ must also be divisible by 8, so a must be even (2).

Case 3: $c = 6$. For a 3-digit number $\underline{ab6}$ to be divisible by 8, 2-digit number $\underline{b6}$ must be divisible by 4. Since 6 is even but not divisible by 4, 2-digit number $\underline{b0}$ must also be even (which is always true) and not divisible by 4, so b must be odd (1, 3, or 5).

Sub-case 3.1: $b = 1$. Consider number $\underline{a16}$. We want it to be divisible by 8. Since 16 is divisible by 8, 3-digit number $\underline{a00}$ must also be divisible by 8, so a must be even (2 or 4).

Sub-case 3.2: $b = 3$. Consider number $\underline{a36}$. We want it to be divisible by 8. Since 36 is divisible by 4 but not divisible by 8, 3-digit number $\underline{a00}$ must also be divisible by 4 (which is always true) and not divisible by 8, so a must be odd (1 or 5).

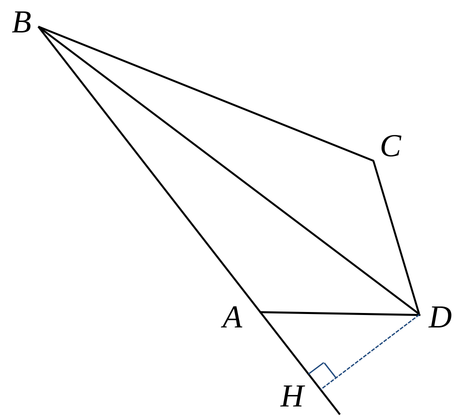
Sub-case 3.3: $b = 5$. Consider number $\underline{a56}$. We want it to be divisible by 8. Since 56 is divisible by 8, 3-digit number $\underline{a00}$ must also be divisible by 8, so a must be even (2 or 4).

So we found that there are 14 different 3-digit "endings" for a 6-digit number on the board to be divisible by 8 (312, 512, 432, 632, 152, 352, 624, 264, 216, 416, 136, 536, 256, 456). Let's consider one of these endings, $\underline{abc}$, and count how many 6-digit numbers on the board end with $\underline{abc}$. For such a 6-digit number $\underline{defabc}$, we have 3 choices for digit d (any of the digits 1, 2, 3, 4, 5, 6, except a, b, c). After we selected d, we have 2 choices for digit e (any of the digits 1, 2, 3, 4, 5, 6, except a, b, c, d). Finally, after we selected d and e, we have just 1 choice for digit f (the remaining digit). So, for each of 14 possible 3-digit endings $\underline{abc}$ we can find exactly $3\times2\times1 = 6$ 6-digit numbers on the board with such an ending. Since the divisibility of a 6-digit number by 8 is equivalent to the divisibility of its 3-digit ending by 8, there are just 14 such 3-digit endings, and each of them is an ending for 6 different 6-digit numbers on the board, we conclude that exactly $6\times14 = 84$ numbers on the board are divisible by 8.

12. Anton took three sides of an equilateral triangle with perimeter 2016, cut one of the sides into two equal parts, and made a quadrilateral which is not a parallelogram using these four segments. What is the greatest possible area of this quadrilateral?

Answer: 225,792 (or 225792)

Solution. Each side of the equilateral triangle is of length $2016\div3 = 672$. Anton cut one of the sides into two equal parts, each of length $672\div2 = 336$. Therefore the four segments are of length 672, 672, 336, and 336. Anton made a quadrilateral using these four segments. If two longer sides (of equal length) of this quadrilateral are opposite to each



other, then the two shorter sides (of equal length) are opposite to each other as well. In this case the quadrilateral is a parallelogram. But the quadrilateral Anton made was not a parallelogram, therefore its two longer sides (of equal length) are adjacent to each other, and its two shorter sides (of equal length) are adjacent to each other as well. Such a quadrilateral is known as a kite (see the diagram). Let's name its vertices A, B, C, and D, where B is the shared vertex of the two longer sides and D is the shared vertex of the two shorter sides. Since $AB = BC$, $AD = DC$, $BD = BD$, we conclude that $\Delta ABD \cong \Delta CBD$, and therefore the area of $ABCD$ is twice the area of ΔABD (let's call these areas X and Y respectively).

Let's draw an altitude DH from D to line AB, where point H is on this line. Since $DH \leq DA$, we can bound the area of ΔABD: $Y = \frac{BA \cdot DH}{2} \leq \frac{BA \cdot DA}{2}$, and therefore $X = 2Y \leq BA \cdot DA = 672 \cdot 336 = 225792$. This value is achieved when $H = A$, i.e. when $m\angle BAD = 90°$.

Note 1. We consider only simple quadrilaterals since the corresponding self-intersecting quadrilaterals have smaller areas. We consider only convex kites since the corresponding concave kites have smaller areas.

Note 2. By cutting a convex kite $ABCD$ along its line of symmetry BD and gluing the two congruent triangles ABD and CBD along segment BD (B to D and D to B) we get a parallelogram with the same side lengths as the kite, and with the same area as the kite. And vice versa, using cutting and gluing we can convert a parallelogram into a convex kite. Thus, the greatest possible area of a kite with side lengths 672, 672, 336, and 336 is the same as the greatest possible area of a parallelogram with side lengths 672, 672, 336, and 336, which is achieved for a rectangle with side lengths 672, 672, 336, and 336. The area of this rectangle is $672 \cdot 336 = 225792$.